Randomization and the Robustness of Linear Contracts

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Motivation

- The standard principal-agent model thinks like a (Bayesian) statistician.

  $\Rightarrow$ (often) complicated contracts tailored to specifics of environment.

- Long history of pursuing foundations for something "simpler".

- Carroll (AER, 2015): in a non-Bayesian model, linear contracts are robustly optimal because they align the principal and agent's interests.

Motivation

- Analysis restricted to study of optimal deterministic contracts.
- Natural to consider randomization in max-min problems.
  - In zero-sum games, randomization can strictly increase minimax payoff.
  - Raiffa (QJE, 1961): randomization can be used to alleviate ambiguity aversion.
- Kambhampati (JET, 2023): randomization strictly benefits the principal.
- What do robustly contracts look like? Are they still linear? Or “simple”?
- **This paper:** Optimal to randomize uniformly over just two linear contracts!

A Robust Principal-Agent Problem

- Principal contracts with agent to produce output in compact set $Y \subset \mathbb{R}_+$.
  - $\min(Y) = 0 < \bar{e} = \max(Y)$.
- Principal *knows* agent can take hidden action $(F_0, c_0) \in \Delta(Y) \times \mathbb{R}_+$.
  - $F_0 \in \Delta(Y)$ is probability distribution over output, with mean e_0.
  - $c_0 \in \mathbb{R}_+$ is effort cost.
  - Assume $e_0 - c_0 > 0$ and $c_0 > 0$.
- True set of hidden actions is a compact set $A \subset \Delta(Y) \times \mathbb{R}_+$ containing a_0.
- Both parties risk-neutral.

A Robust Principal-Agent Problem

- A (deterministic) **contract** is a cts function $w : Y \to \mathbb{R}$.
  - Bilateral limited liability: $0 \leq w(y) \leq y$ for all $y \in Y$.
  - Participation constraint: $E_{F_0}[w(y)] - c_0 \geq \bar{u} \geq 0$ (talk only).
- Set of contracts W, endowed with sup-norm topology.
- A **random contract** is a (Borel) probability measure over contracts, $p \in \Delta(W)$.
- Timing:
  1. Principal commits to a contract p.
  2. Nature, knowing p, chooses A.
  3. Agent, knowing w and A, chooses $a = (F, c) \in A$.
  4. Output y realized.
     - Payoff P: $y - w(y)$
     - Payoff A: $w(y) - c$.

Principal's Payoff Guarantee

- Given (w, A), set of optimal actions for agent:

$$B(w, A) := \arg\max_{(F,c)\in A} \mathbb{E}_F[w(y)] - c.$$

- Payoff for principal under (w, A):

$$V(w, A) := \min_{(F,c)\in B(w,A)} \mathbb{E}_F[y - w(y)].$$

- Payoff guarantee for principal under random contract p:

$$V(p) := \inf_{A \ni a_0} \mathbb{E}_p \left[V(w, A) \right].$$

- A random contract is **optimal** if $V(p^*) = \sup_{p\in\Delta(W)} V(p)$.

The Result

- A contract $w \in W$ is *linear* if there exists $\alpha \in [0,1]$ such that $w(y) = \alpha y$.
- A random contract $p \in \Delta(W)$ is *linear* if every contract in its support is linear.

Theorem

There exists an optimal random contract, p, that is linear and has binary support, $\{\alpha_1, \alpha_2\}$. In any such contract, $p(\{\alpha_1\}) = p(\{\alpha_2\}) = \frac{1}{2}$ and $\alpha_1 < \alpha_D < \alpha_2$.

Three steps for today:

1. Any random contract can be improved upon by a linear random contract.
2. There exists an optimal random linear contract.
3. Enough to randomize over two linear contracts .

Proof Sketch: Step 1

- Let $q \in \Delta(W)$ be a random contract.
- Let $T : W \to W$ be a cts linear transformation associating each contract w with a linear contract with slope

$$\alpha_w := \frac{\mathbb{E}_{F_0}[w(y)]}{e_0}.$$

- Define a linear random contract

$$p(B) := q(T^{-1}(B)) \quad \forall \text{ Borel } B \subset W.$$

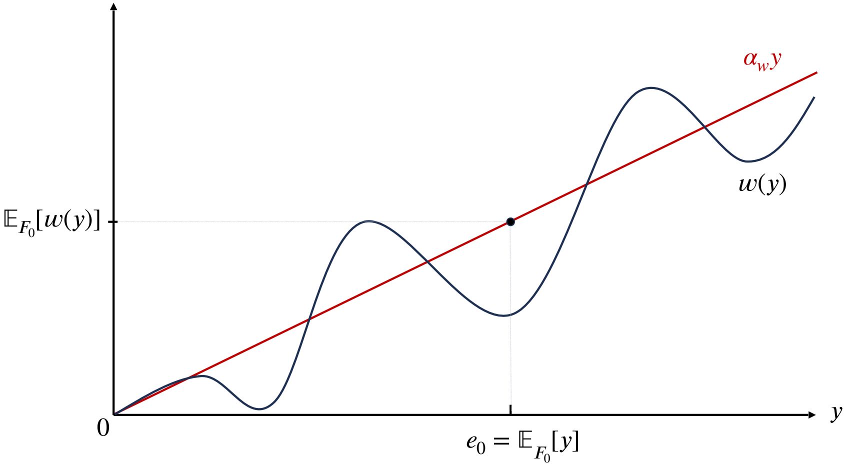
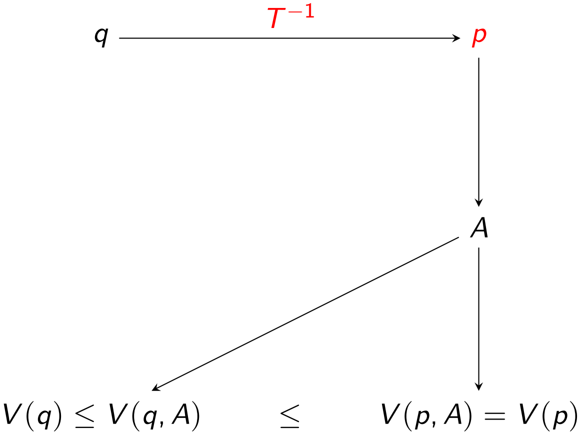


Figure 1: Illustration of the linear transformation $T(\cdot)$.

Proof Sketch: Step 1

Claim: $V(p) \geq V(q)$.



Proof Sketch: Step 1

Associate with any *linear* random contract $p \in \Delta(W)$, the cdf $G_p : [0,1] \to [0,1]$.

$$V(p) = \min_{(e(\alpha),c(\alpha))_{\alpha\in[0,1]}} \int_0^1 (1-\alpha)e(\alpha)\, dG_p(\alpha) \tag{LP(p)}$$

subject to

$$\alpha e(\alpha) - c(\alpha) \geq \alpha e(\alpha') - c(\alpha') \qquad \forall \alpha, \alpha' \in [0,1] : \alpha \neq \alpha', \tag{IC}$$

$$\alpha e(\alpha) - c(\alpha) \geq \alpha e_0 - c_0 \qquad \forall \alpha \in [0,1], \tag{IR}$$

$$c(\alpha) \geq 0,\ 0 \leq e(\alpha) \leq \bar{e} \qquad \forall \alpha \in [0,1]. \tag{F}$$

Analogy to "standard" mechanism design:

- G_p is the distribution over types $\alpha \in [0,1]$.
- $e(\cdot)$ is the allocation rule.
- $c(\cdot)$ is the transfer rule.

Proof Sketch: Step 1

$$V(p) = \min_{e(\cdot)} \int_0^1 (1-\alpha)e(\alpha)\, dG_p(\alpha) \tag{LP(p)}$$

subject to

$e(\cdot)$ is nondecreasing,

$$\int_0^\alpha e(t)dt \geq \alpha e_0 - c_0 \qquad \forall \alpha \in [0,1],$$

$e(0) \geq 0,\ e(1) \leq \bar{e}.$

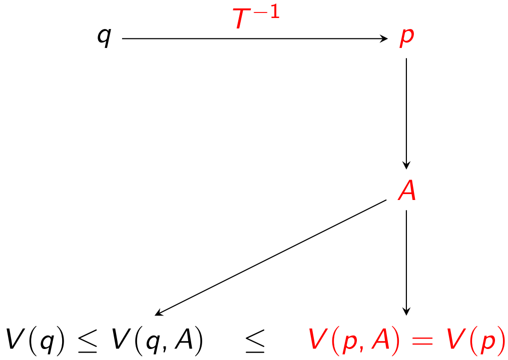
Lemma

There exists a minimizer $e^*(\cdot)$ bounded above by e_0.

Proof Sketch: Step 1

Solution identifies a family of worst-case technologies of the form

$$\text{cl}\left(\{(F_0, c_0)\} \cup \{(F(\alpha), c^*(\alpha))_{\alpha\in[0,1]}\}\right) \qquad \text{with } E_{F(\alpha)}[y] = e^*(\alpha) \le e_0.$$



Proof Sketch: Step 1

Choose a selection from this family that makes q perform poorly:

$$A := \text{cl}\left(\{(F_0, c_0)\} \cup \{(F(\alpha), c^*(\alpha))_{\alpha\in[0,1]}\}\right),$$

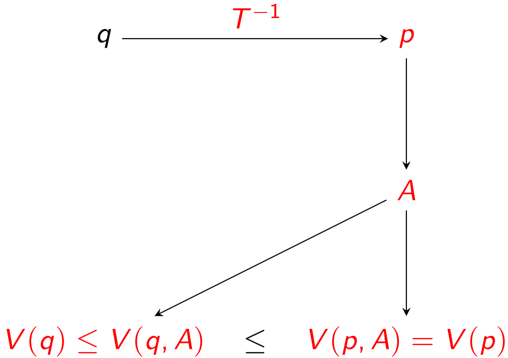
where

$$F(\alpha) := \left(\frac{e^*(\alpha)}{e_0}\right) F_0 + \left(1 - \frac{e^*(\alpha)}{e_0}\right)\delta_0.$$

Notice:

$$\underbrace{\mathbb{E}_{F(\alpha)}[w(y)] = \left(\frac{e^*(\alpha)}{e_0}\right)\mathbb{E}_{F_0}[w(y)] = \alpha_w e^*(\alpha)}_{\Rightarrow \text{ IC satisfied}} \text{ and } \underbrace{\mathbb{E}_{F(\alpha)}[y] = \left(\frac{e^*(\alpha)}{e_0}\right) e_0 = e^*(\alpha)}_{\Rightarrow V(q,A) \le V(p,A)}.$$

Proof Sketch: Step 1



Proof Sketch

- A contract $w \in W$ is *linear* if there exists $\alpha \in [0,1]$ such that $w(y) = \alpha y$.
- A random contract $p \in \Delta(W)$ is *linear* if every contract in its support is linear.

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There exists an optimal random contract, p, that is linear and has binary support, $\{\alpha_1, \alpha_2\}$. In any such contract, $p(\{\alpha_1\}) = p(\{\alpha_2\}) = \frac{1}{2}$ and $\alpha_1 < \alpha_D < \alpha_2$.

Three steps:

1. Any random contract can be improved upon by a linear random contract. ✓
2. There exists an optimal random linear contract.
3. Enough to randomize over two linear contracts.

Proof Sketch: Step 2

- Suffices to check whether there is a contract that maximizes

$$V(p) = \min_{e(\cdot)} \int_0^1 (1-\alpha) e(\alpha)\, dG_p(\alpha) \tag{LP(p)}$$

$$\text{subject to}$$

$$e(\cdot) \text{ is nondecreasing,}$$

$$\int_0^\alpha e(t)dt \geq \alpha e_0 - c_0 \qquad \forall \alpha \in [0,1],$$

$$e(0) \geq 0, \ e(1) \leq \bar{e}.$$

- If $V(\cdot)$ is continuous (in the topology of weak convergence), then existence follows from compactness of $\Delta([0,1])$.

Proof Sketch: Step 2

Lemma

$p \mapsto V(p)$ is a continuous map from $\Delta([0,1])$ to $\mathbb{R}$.

Proof Sketch:

- Let $V_k(p)$ be P's payoff when Nature's choice $e(\cdot)$ is k-Lipschitz continuous.
- Feasible set compact in sup-norm topology (Arzelà-Ascoli).
- Objective function becomes continuous in $(e(\cdot), p)$.
- So $V_k(\cdot)$ is continuous by Maximum Theorem.
- Sequence (V_k) converges uniformly to V, establishing its continuity.

Proof Sketch

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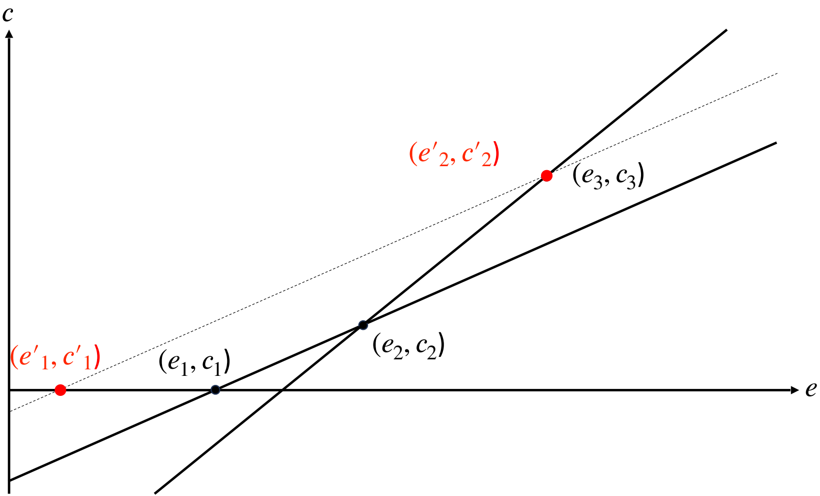
Proof Sketch: Step 3

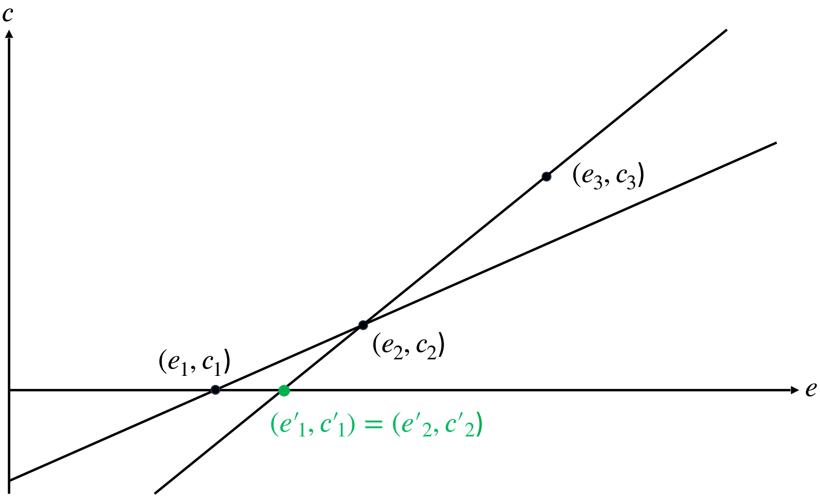
- Because $V(\cdot)$ is continuous and finite random contracts are dense in $\Delta([0,1])$, suffices to establish improvement argument for linear random contracts with finite support.
- Will utilize (another) important lemma:

Lemma

Let p be a linear random contract with $\text{supp}(p) = \{\alpha_1, \dots, \alpha_I\}$ and probabilities $(p_i)_i$. Then, LP(p) has a solution $(e_i, c_i)_{i=1}^{I}$ such that $\#\{e_i : i \in [1, I]\} \leq 2$.



c
e
(e'_2, c'_2)
(e_3, c_3)
(e_2, c_2)
(e'_1, c'_1)
(e_1, c_1)



c
e
(e_3, c_3)
(e_2, c_2)
(e_1, c_1)
$(e'_1, c'_1) = (e'_2, c'_2)$

Proof Sketch: Step 3

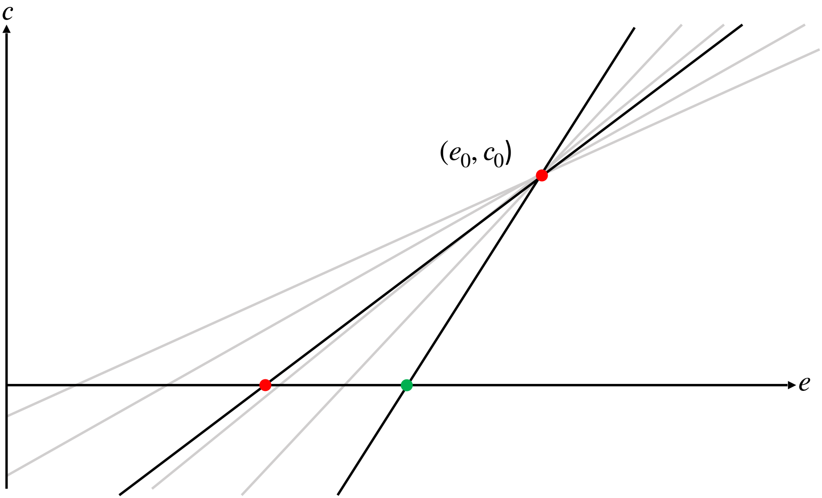
- P's payoff:

$$V(p) = \min_{k \in [1, I]} \quad \sum_{i=1}^{k} p_i (1 - \alpha_i) \left(e_0 - \frac{c_0}{\alpha_k} \right) + \sum_{i=k+1}^{I} p_i (1 - \alpha_i) e_0 .$$

- If probabilities chosen optimally, then at most one type takes the known action $(k \geq I - 1)$. So:

$$V(p) = \min \{ \underbrace{\sum_{i=1}^{I} p_i (1 - \alpha_i) \left(e_0 - \frac{c_0}{\alpha_I} \right)}_{\text{Pooling}}, \underbrace{\sum_{i=1}^{I-1} p_i (1 - \alpha_i) \left(e_0 - \frac{c_0}{\alpha_{I-1}} \right)}_{\text{Pooling}} + p_I (1 - \alpha_I) e_0 \}$$

- Collapse pooling region into a single contract played with prob 1 or $\sum_{i=1}^{I-1} p_i$.



c
(e_0, c_0)
e

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2. There exists an optimal linear contract. ✓
3. Enough to randomize over two linear contracts. ✓

Final Remarks

- Randomization strictly benefits the principal in robust moral hazard problems.
- Nevertheless, optimal random contract is still linear and "simple".
- Other extensions:
  - Results go through without participation constraint.
  - Screening doesn't help.
  - Value of randomization is unbounded.

Thank you!